

# Verma modules over the generalized Heisenberg-Virasoro algebra<sup>1</sup>

Ran Shen\*, Yucai Su<sup>†</sup>

*\*Department of Mathematics, Shanghai Jiao Tong University  
Shanghai 200240, China*

*<sup>†</sup>Department of Mathematics, University of Science and Technology of China  
Hefei 230026, China*

*Email: ranshen@sjtu.edu.cn, ycsu@ustc.edu.cn*

**Abstract.** For any additive subgroup  $G$  of an arbitrary field  $\mathbb{F}$  of characteristic zero, there corresponds a generalized Heisenberg-Virasoro algebra  $\mathcal{L}[G]$ . Given a total order of  $G$  compatible with its group structure, and any  $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$ , a Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  over  $\mathcal{L}[G]$  is defined. In this note, the irreducibility of Verma modules  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is completely determined.

**Key Words:** The generalized Heisenberg-Virasoro algebra, Verma modules

*Mathematics Subject Classification (2000):* 17B56; 17B68.

## 1. Introduction

Let  $\mathbb{F}$  be a field of characteristic 0. The well-known *twisted Heisenberg-Virasoro algebra* is the Lie algebra  $\mathcal{L} := \mathcal{L}[\mathbb{Z}]$  with an  $\mathbb{F}$ -basis  $\{L_m, I_m, C, C_I, C_{LI} \mid m \in \mathbb{Z}\}$  subject to the following relations (e.g., [ACKP, B])

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C,$$

$$[L_n, I_m] = -mI_{n+m} - \delta_{n,-m}(n^2 + n)C_{LI},$$

$$[I_n, I_m] = n\delta_{n,-m}C_I,$$

$$[\mathcal{L}, C] = [\mathcal{L}, C_{LI}] = [\mathcal{L}, C_I] = 0.$$

This Lie algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one, which contains an infinite-dimensional Heisenberg subalgebra and the Virasoro subalgebra. The natural action of the Virasoro subalgebra on the Heisenberg subalgebra is twisted with a 2-cocycle. The structure and representation theory for the twisted Heisenberg-Virasoro algebra has been well developed (e.g., [ACKP, B, FO, JJ, SJ]). The structure of the irreducible highest weight modules for the twisted Heisenberg-Virasoro algebra are determined in [ACKP, B].

By replacing the index group  $\mathbb{Z}$  by an arbitrary subgroup  $G$  of the base field  $\mathbb{F}$ , it is

---

<sup>1</sup>Supported by NSF grants 10471096, 10571120 of China and “One Hundred Talents Program” from University of Science and Technology of China

natural to introduce the so-called *generalized Heisenberg-Virasoro algebra*  $\mathcal{L}[G]$  (cf. Definition 2.1, see e.g., [XLT, LJ]). This is the Lie algebra which is the 3-dimensional universal central extension of the Lie algebra of generalized differential operators of order at least one. The Harish-Chandra modules of intermediate series over generalized Heisenberg-Virasoro algebra  $\mathcal{L}[G]$  are discussed in [LJ].

Given any total order of  $G$  compatible with its group structure, and given any  $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$ , there corresponds a Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  over  $\mathcal{L}[G]$ . Due to the fact that the representations of generalized Heisenberg-Virasoro algebras are closely related to the representation theory of toroidal Lie algebras as well as some problems in mathematical physics (e.g., [ACKP, FO, JJ]) and the Verma modules play the crucial role in the representation theory, it is very natural to consider the Verma modules over the generalized Heisenberg-Virasoro algebras. In this note, we completely determine the irreducibility of Verma modules  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  over  $\mathcal{L}[G]$  for arbitrary  $G$ . Namely, if  $G$  does not contain a minimal positive element with respect to the total order, then the Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is irreducible if and only if  $(c_I, c_{LI}) \neq (0, 0)$ ; in case if  $G$  contains the minimal positive element  $a$ , then the Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is irreducible if and only if the  $\mathcal{L}[\mathbb{Z}a]$ -module generated by a fixed highest weight generator is irreducible over the twisted Heisenberg-Virasoro algebra  $\mathcal{L}[\mathbb{Z}a]$  (cf. Theorem 3.1).

## 2. Generalized Heisenberg-Virasoro algebras

Let  $U := U(\mathcal{L})$  be the universal enveloping algebra of the twisted Heisenberg-Virasoro algebra  $\mathcal{L}$ . For any  $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$ , denote by  $I(h, h_I, c, c_I, c_{LI})$  the left ideal of  $U$  generated by the elements

$$\{L_i, I_j \mid i, j > 0\} \cup \{L_0 - h \cdot 1, I_0 - h_I \cdot 1, C - c \cdot 1, C_I - c_I \cdot 1, C_{LI} - c_{LI} \cdot 1\}.$$

The *Verma module* with highest weight  $(h, h_I, c, c_I, c_{LI})$  over  $\mathcal{L}$  is defined as

$$M(h, h_I, c, c_I, c_{LI}) := U/I(h, h_I, c, c_I, c_{LI}),$$

which is a highest weight module with a basis consisting of all vectors of the form

$$I_{-p_1} I_{-p_2} \cdots I_{-p_s} L_{-j_1} L_{-j_2} \cdots L_{-j_k} v_h, \quad (2.1)$$

where  $s, k \in \mathbb{N} \cup \{0\}$ ,  $p_r, j_i \in \mathbb{N}$  and  $0 < p_1 \leq p_2 \leq \cdots \leq p_s$ ,  $0 < j_1 \leq j_2 \leq \cdots \leq j_k$ .

**Definition 2.1** Let  $G \subseteq \mathbb{F}$  be an additive subgroup. The *generalized Heisenberg-Virasoro algebra*  $\widetilde{\mathcal{L}} := \mathcal{L}[G]$  is a Lie algebra with  $\mathbb{F}$ -basis  $\{L_\mu, I_\mu, C, C_I, C_{LI} \mid \mu \in G\}$  subject

to the following relations [XLT, LJ]

$$\begin{aligned}
[L_\mu, L_\nu] &= (\mu - \nu)L_{\mu+\nu} + \delta_{\mu,-\nu} \frac{\mu^3 - \mu}{12} C, \\
[L_\mu, I_\nu] &= -\nu I_{\mu+\nu} - \delta_{\mu,-\nu} (\mu^2 + \mu) C_{LI}, \\
[I_\mu, I_\nu] &= \mu \delta_{\mu,-\nu} C_I, \\
[\tilde{\mathcal{L}}, C] &= [\tilde{\mathcal{L}}, C_{LI}] = [\tilde{\mathcal{L}}, C_I] = 0.
\end{aligned}$$

For any  $x \in G^* := G \setminus \{0\}$ , obviously,  $\mathbb{Z}x \subseteq G$ . Let  $\mathcal{L}[\mathbb{Z}x]$  be the  $\mathbb{F}$ -subspace of  $\tilde{\mathcal{L}}$  spanned by  $\{L_{ix}, I_{ix}, C, C_I, C_{LI} \mid i \in \mathbb{Z}\}$ . It is clear that  $\mathcal{L}[\mathbb{Z}x]$  is a Lie algebra isomorphic to the twisted Heisenberg-Virasoro algebra  $\mathcal{L}$ . Precisely, we have

**Lemma 2.2** The map

$$\begin{aligned}
\theta : \mathcal{L} &\rightarrow \mathcal{L}[\mathbb{Z}x] \\
L_i &\mapsto x^{-1}L_{ix} + \delta_{i,0} \frac{x-x^{-1}}{24} C, \\
I_i &\mapsto x^{-1}I_{ix} + \delta_{i,0} (1 - x^{-1}) C_{LI}, \\
C &\mapsto xC, \\
C_I &\mapsto x^{-1}C_I, \\
C_{LI} &\mapsto C_{LI},
\end{aligned}$$

for  $i \in \mathbb{Z}$ , extends uniquely to a Lie algebra isomorphism between  $\mathcal{L}$  and  $\mathcal{L}[\mathbb{Z}x]$ .

*Proof.* This follows from straightforward verifications.  $\square$

Throughout this note, we fix a total order “ $\succ$ ” on  $G$  compatible with its group structure, namely,  $x \succ y$  implies  $x + z \succ y + z$  for any  $z \in G$ . Denote

$$G_+ := \{x \in G \mid x \succ 0\}, \quad G_- := \{x \in G \mid x \prec 0\}.$$

Then  $G = G_+ \cup \{0\} \cup G_-$ .

For an  $\tilde{\mathcal{L}}$ -module  $V$  and  $\lambda, h_I, c, c_I, c_{LI} \in \mathbb{F}$ , denote by

$$V_{\lambda, h_I, c, c_I, c_{LI}} := \{v \in V \mid L_0 v = \lambda v, I_0 v = h_I v, C v = c v, C_I v = c_I v, C_{LI} v = c_{LI} v\},$$

the *weight space* of  $V$ . We shall simply write  $V_\lambda$  instead of  $V_{\lambda, h_I, c, c_I, c_{LI}}$ . Define

$$\text{supp}(V) := \{\lambda \in \mathbb{F} \mid V_\lambda \neq 0\},$$

called the *weight set* (or the *support*) of  $V$ . For any  $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$ , let  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  be the *Verma module* for  $\tilde{\mathcal{L}}$ , which is defined by using the order “ $\succ$ ” and the same fashion

as that for  $\mathcal{L}$  at the beginning of this section. Then  $I_0, C, C_I, C_{LI}$  acts as  $h_I, c, c_I, c_{LI}$  respectively on  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  and

$$\text{supp}(\widetilde{M}(h, h_I, c, c_I, c_{LI})) = h + G_+.$$

For any  $x \in G_+$ , let

$$\widetilde{M}_x(h, h_I, c, c_I, c_{LI}) = U(\mathcal{L}[\mathbb{Z}x])v_h, \quad (2.2)$$

be the  $\mathcal{L}[\mathbb{Z}x]$ -submodule of  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  generated by a fixed highest weight generator  $v_h$ . Note that the subgroup  $\mathbb{Z}x$  is also a “totally ordered abelian group”, inheriting the order “ $\succ$ ” from  $G$ . It is easy to see that

$$ax \succ bx \iff a > b \text{ for } a, b \in \mathbb{Z}.$$

As a result, we have

**Corollary 2.3** As an  $\mathcal{L}$ -module, we have

$$\widetilde{M}_x(h, h_I, c, c_I, c_{LI}) \cong M(x^{-1}h + \frac{x - x^{-1}}{24}c, x^{-1}h_I + (1 - x^{-1})c_{LI}, xc, x^{-1}c_I, c_{LI}).$$

*Proof.* This is clear by Lemma 2.2. □

### 3. The main result

Recall that  $(G, \succ)$  is a totally ordered abelian group. Denote

$$B(x) = \{y \in G \mid 0 \prec y \prec x\} \text{ for } x \in G_+.$$

The order “ $\succ$ ” is called *dense* if  $\#B(x) = \infty$  for all  $x \in G_+$ ; *discrete* if there exists some  $a \in G_+$  such that  $B(a) = \emptyset$ , in this case  $a$  is called the *minimal positive element* of  $G$ .

For convenience, we denote

$$L_{-j} := L_{-j_1}L_{-j_2} \cdots L_{-j_k} \text{ for } 0 \prec j_1 \preceq j_2 \preceq \cdots \preceq j_k, \ j = (j_1, j_2, \cdots, j_k), \quad (3.1)$$

$$I_{-p} := I_{-p_s}I_{-p_{s-1}} \cdots I_{-p_1} \text{ for } 0 \prec p_s \preceq \cdots \preceq p_2 \preceq p_1, \ p = (p_s, \cdots, p_2, p_1). \quad (3.2)$$

Then  $U(\widetilde{\mathcal{L}}_-)$  has a basis

$$\{I_{-p}L_{-j} \mid \text{for all } j, p \text{ as in (3.1) and (3.2)}\}. \quad (3.3)$$

Denote by  $|j|$  the number of components in  $j$ . Then  $|j| = k$  in (3.1) and  $|p| = s$  in (3.2).

The main result in this note is following.

**Theorem 3.1** Let  $h, h_I, c, c_I, c_{LI} \in \mathbb{F}$ .

- (1) With respect to a dense order “ $\succeq$ ” of  $G$ , the Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is an irreducible  $\mathcal{L}[G]$ -module if and only if  $(c_I, c_{LI}) \neq (0, 0)$ .

- (2) With respect to a discrete order “ $\succeq$ ” of  $G$  with minimal positive element  $a$ , the Verma module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is an irreducible  $\mathcal{L}[G]$ -module if and only if  $\widetilde{M}_a(h, h_I, c, c_I, c_{LI})$  (cf. (2.2)) is an irreducible  $\mathcal{L}[\mathbb{Z}a]$ -module.

**Remark 3.2** Suppose  $c_I = c_{LI} = 0$  in case of Theorem 3.1(1). Since

$$\widetilde{\mathcal{I}} := \text{span}_{\mathbb{F}}\{I_{\mu}, C_I, C_{LI} \mid \mu \in G\},$$

is an ideal of  $\widetilde{\mathcal{L}}$ , the Verma module  $V := \widetilde{M}(h, h_I, c, 0, 0)$  over  $\widetilde{\mathcal{L}}$  has a proper submodule  $U(\widetilde{\mathcal{I}})V$  such that the quotient module  $W := V/U(\widetilde{\mathcal{I}})V$  is simply the Verma module over the generalized Virasoro algebra  $\text{Vir}[G] := \text{span}_{\mathbb{F}}\{L_{\mu}, C \mid \mu \in G\} \cong \widetilde{\mathcal{L}}/\widetilde{\mathcal{I}}$ , whose irreducibility is completely determined in [HWZ]. Also note that the irreducibility of a Verma module over the twisted Heisenberg-Virasoro algebra  $\mathcal{L}$  is completely determined in [B]. Thus, essentially the above theorem has in fact determined the structure of all Verma modules over  $\widetilde{\mathcal{L}}$ .

*Proof of Theorem 3.1.* (1) Suppose the order “ $\succeq$ ” of  $G$  is dense. Let  $v_h$  be a fixed highest weight generator in  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  of weight  $h$ . Let  $u_0 \notin \mathbb{F}v_h$  be any given weight vector in  $V := \widetilde{M}(h, h_I, c, c_I, c_{LI})$ .

**Claim 1:** There exists a weight vector  $u \in U(\mathcal{L}[G])u_0$  of weight  $\lambda$  such that

$$u = \sum_p a_p I_{-p} v_h \text{ (a finite sum) for some } a_p \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}. \quad (3.4)$$

For each  $m \in \mathbb{N}$ , set

$$V_m := \sum_{p, j: |j| \leq m} \mathbb{F} I_{-p} L_{-j} v_h. \quad (3.5)$$

It is clear that

$$L_x V_m \subseteq V_m, \quad I_x V_m \subseteq V_m \text{ for } x \in G_+.$$

We can write  $u_0$  as (cf. (2.1) and (3.3))

$$u_0 = \sum_{p, j} a_{pj} I_{-p} L_{-j} v_h \text{ for some } a_{pj} \in \mathbb{F}^*.$$

Let  $r := \max \{|j| \mid a_{pj} \neq 0\}$ . If  $r = 0$ , then the claim holds clearly. We assume  $r \geq 1$ , and write

$$u_0 \equiv u'_0 \pmod{V_{r-1}}, \text{ where } u'_0 = \sum_{p, j: |j|=r} a_{pj} I_{-p} L_{-j} v_h. \quad (3.6)$$

Let  $x \in G_+$  such that (cf. (3.2) for notation  $p_l$ )

$$x \prec \min\{j_1 \mid a_{pj} \neq 0\} \text{ and } \{x, j_1 - x \mid a_{pj} \neq 0\} \cap \{p_l \mid a_{pj} \neq 0, \forall l\} = \emptyset.$$

Then

$$I_x u'_0 = \sum_{p,j: |j|=r} x a_{pj} I_{-p} \left( \sum_{i=1}^r L_{-j_1} \cdots L_{-j_{i-1}} I_{x-j_i} L_{-j_{i+1}} \cdots L_{-j_r} \right) v_h.$$

If any

$$x a_{pj} I_{-p} L_{-j_1} \cdots L_{-j_{i-1}} I_{x-j_i} L_{-j_{i+1}} \cdots L_{-j_r} \text{ and } x a_{p'j'} I_{-p'} L_{-j'_1} \cdots L_{-j'_{s-1}} I_{x-j'_s} L_{-j'_{s+1}} \cdots L_{-j'_r},$$

for  $1 \leq i, s \leq r$ , are linear dependent, it is not difficult to see that  $p = p'$  and  $j = j'$ . Hence

$$0 \neq u_1 := I_x u'_0 \in V_{r-1}.$$

Similarly, let  $u_1 \equiv u'_1 \pmod{V_{r-2}}$  as in (3.6), then  $u'_1 \neq 0$ . For  $k = 2, \dots, r$ . We define recursively and prove by induction that,

$$u_k := I_x u_{k-1} \in V_{r-k}, \quad u_k \equiv u'_k \pmod{V_{r-k-1}}, \quad u'_k \neq 0.$$

Letting  $k = r$ , we get that  $0 \neq u_r \in V_0$ . Our claim follows.

Now let  $u$  be as in (3.4). Set  $P := \{p \mid a_p \neq 0\} \neq \emptyset$ . We define the total order " $\succ$ " on  $P$  as follows: For any  $p, p' \in P$ , if  $k := |p| > l := |p'|$ , we set  $p'_i = 0$  for  $i = l + 1, \dots, k$ . Then

$$p \succ p' \iff \exists s \text{ with } 1 \leq s \leq k \text{ such that } p_s \succ p'_s \text{ and } p_t = p'_t \text{ for } t < s. \quad (3.7)$$

Let

$$q := (q_{k_0}, \dots, q_2, q_1), \quad 0 \prec q_{k_0} \preceq \cdots \preceq q_1,$$

be the unique maximal element in  $P$ . Then

Case 1: If  $c_I \neq 0$ , then by the simple calculations

$$b v_h = I_q u \in U(\mathcal{L}[G]) u_0 \text{ for some } b \in \mathbb{F}^*.$$

Case 2: Suppose  $c_I = 0$ ,  $c_{LI} \neq 0$ . Let  $y \in G_+$  such that

$$\{x \in G \mid q_1 - y \prec x \prec q_2\} \cap \{p_1, p_2 \mid p \in P\} = \emptyset.$$

Then

$$u' := L_{q_1-y} u = a' I_{-z} v_h \text{ for some } a' \in \mathbb{F}^*,$$

where

$$z = (z_{k_0}, \dots, z_2, z_1), \quad 0 \prec z_{k_0} \preceq \cdots \preceq z_2 \preceq z_1, \quad \text{and} \\ \{z_i \mid i = 1, 2, \dots, k_0\} = \{q_{k_0}, \dots, q_3, q_2, y\}.$$

(i) If  $\{z_i \mid i = 1, 2, \dots, k_0\} \cap \{h_I / c_{LI} - 1\} = \emptyset$ , then

$$b' v_h = L_z u' \in U(\mathcal{L}[G]) u_0 \neq 0, \text{ where } b' = \prod_{i=1}^{k_0} z_i (h_I - (z_i + 1) c_{LI}) \in \mathbb{F}^*.$$

(ii) If there exists some  $z_i = h_I/c_{LI} - 1$  with  $1 \leq i \leq k_0$ . We assume

$$\{z_i\} \cap \{z_k \mid 1 \leq k \leq k_0, k \neq i\} = \emptyset.$$

Otherwise, we only need to recurse the following proof. Let

$$w := L_{z_{i-1}} \cdots L_{z_2} L_{z_1} u' = a'' I_{-z_{k_0}} I_{-z_{k_0-1}} \cdots I_{-z_i} v_h \neq 0 \text{ for some } a'' \in \mathbb{F}^*.$$

Take  $x' \in G_+$  such that  $z_i - x' \succ z_k, i < k \leq k_0$ . Then

$$w' := L_{z_i - x'} w = \bar{a} I_{-z_{k_0}} I_{-z_{k_0-1}} \cdots I_{-z_{i+1}} I_{-x'} v_h \neq 0 \text{ for some } \bar{a} \in \mathbb{F}^*,$$

and  $\{z_{k_0}, z_{k_0-1}, \dots, z_{i+1}, x'\} \cap \{h_I/c_{LI} - 1\} = \emptyset$ . This becomes case (i) if we take  $u'$  to be  $w'$ .

Therefore,  $v_h \in U(\mathcal{L}[G])u_0$  in any case. Hence  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  is irreducible.

(2) Suppose the order “ $\succ$ ” of  $G$  is discrete with the minimal positive element  $a$ . Then  $\mathbb{Z}a \subseteq G$ . For any  $x \in G$ , we write  $x \succ \mathbb{Z}a$  if  $x \succ na$  for all  $n \in \mathbb{Z}$ . Let

$$H_+ := \{x \in G \mid x \succ \mathbb{Z}a\}, \quad H_- = -H_+.$$

It is not difficult to see that

$$G = \mathbb{Z}a \cup H_+ \cup H_-. \quad (3.8)$$

Then one can see that

$$\mathcal{L}[H_+] \widetilde{M}_a(h, h_I, c, c_I, c_{LI}) = 0 \quad (\text{recall (2.2)}).$$

Since

$$\widetilde{M}(h, h_I, c, c_I, c_{LI}) \cong U(\mathcal{L}[G]) \otimes_{U(\mathcal{L}[\mathbb{Z}a] + \mathcal{L}[H_+])} \widetilde{M}_a(h, h_I, c, c_I, c_{LI}),$$

it follows that the irreducibility of  $\mathcal{L}[G]$ -module  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  imply the irreducibility of  $\mathcal{L}[\mathbb{Z}a]$ -module  $\widetilde{M}_a(h, h_I, c, c_I, c_{LI})$ .

Conversely, suppose  $\widetilde{M}_a(h, h_I, c, c_I, c_{LI})$  is an irreducible  $\mathcal{L}[\mathbb{Z}a]$ -module. Let  $u_0 \notin \mathbb{F}v_h$  be any weight vector in  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$ . We want to prove

$$U(\mathcal{L}[G])u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI}) \neq \{0\}, \quad (3.9)$$

from which the irreducibility of  $\widetilde{M}(h, h_I, c, c_I, c_{LI})$  as  $\mathcal{L}[G]$ -module follows immediately.

Case 1:  $c_I \neq 0$ . We can write  $u_0$  as (cf. (3.8))

$$u_0 \equiv \sum_{p'_l, j'_k \in H_+, p_s, j_r \in \mathbb{Z}a, |j'| + |j| = r} a_{p'j'pj} I_{-p'} L_{-j'} I_{-p} L_{-j} v_h \pmod{V_{r-1}} \text{ for some } a_{p'j'pj} \in \mathbb{F}^*,$$

where  $V_{r-1}$ ,  $r$  are defined as in (3.5) and (3.6). Let (cf. notation (3.2))

$$P' = \{pp' = (p_s, \dots, p_2, p_1, p'_l, \dots, p'_2, p'_1) \mid 0 \prec p_s \preceq \dots \preceq p_2 \preceq p_1 \prec p'_l \preceq \dots \preceq p'_2 \preceq p'_1, a_{p'j'pj} \neq 0\}.$$

If  $P' \neq \emptyset$ , we define the total order “ $\succ$ ” on  $P'$  as in (3.7). Let  $q^0$  be the maximal element in  $P'$ . Then

$$u'_0 := I_{q^0} u_0 \equiv \sum_{j'_k \in H_+, j_r \in \mathbb{Z}_+ a, |j'| + |j| = r} a_{j'j} L_{-j'} L_{-j} v_h \pmod{V_{r-1}} \text{ for some } a_{j'j} \in \mathbb{F}^*.$$

If  $P' = \emptyset$ , then  $u_0$  has the form of  $u'_0$  naturally. By the proof of [HWZ, Theorem 3.1], there exists a weight vector  $0 \neq u \in U(\mathcal{L}[G])u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI})$ , which gives (3.9) as required.

Case 2:  $c_I = 0$ . We can write

$$u_0 = \sum_{p'_l, j'_k \in H_+, p_s, j_r \in \mathbb{Z}_+ a} b_{p'j'pj} I_{-p'} L_{-j'} I_{-p} L_{-j} v_h \text{ for some } b_{p'j'pj} \in \mathbb{F}^*.$$

If  $J := \{j' \mid b_{p'j'pj} \neq 0\} \neq \emptyset$ , we set  $j(0) := \min\{j'_1 \mid b_{p'j'pj} \neq 0\}$ . Then there exists some  $m \in \mathbb{N}$  such that

$$\{j'_1 - \varepsilon \mid b_{p'j'pj} \neq 0\} \cap \{p'_l \mid b_{p'j'pj} \neq 0, \forall l\} = \emptyset, \text{ where } \varepsilon = j(0) - ma.$$

Let  $n_0 = \max\{|j'| \mid b_{p'j'pj} \neq 0\}$ , then

$$u' := I_\varepsilon^{n_0} u_0 = \sum_{p'_l \in H_+, p_s, j_r \in \mathbb{Z}_+ a} b'_{p'pj} I_{-p'} I_{-p} L_{-j} v_h \neq 0 \text{ for some } b'_{p'pj} \in \mathbb{F}^*,$$

by the proof of Claim 1. If  $J = \emptyset$ , then  $u_0$  has the form of  $u'$  naturally. Let

$$Q := \{p' \mid b'_{p'pj} \neq 0, |p'| = t\}, \text{ where } t = \min\{|p'| \mid b'_{p'pj} \neq 0\}.$$

If  $t = 0$ , the theorem holds clearly since  $u'$  is a weight vector. We assume  $t \geq 1$ . Then  $Q \neq \emptyset$ . Again, we define the total order “ $\prec$ ” on  $Q$  as in (3.7). Let

$$q' := (q'_1, q'_2, \dots, q'_t), \quad 0 \prec q'_1 \preceq q'_2 \preceq \dots \preceq q'_t,$$

be the unique minimum element in  $Q$ . For  $m \in \mathbb{N}$ , set

$$V'_m := \sum_{p'_l \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p'| \geq m} \mathbb{F} I_{-p'} I_{-p} L_{-j} v_h.$$

Then

$$u' \equiv \sum_{p'_l \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p'| = t} b'_{p'pj} I_{-p'} I_{-p} L_{-j} v_h \pmod{V'_{t+1}}.$$



We have

$$u(1) := L_{q'_1 - a} u' \equiv \sum_{p_i^{(1)} \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p^{(1)}| = t-1} b_{p^{(1)} p_j}^{(1)} I_{-p^{(1)}} I_{-a} I_{-p} L_{-j} v_h \pmod{V'_t}$$

for some  $b_{p^{(1)} p_j}^{(1)} \in \mathbb{F}^*$ . Define  $Q^{(1)} = \{p^{(1)} \mid b_{p^{(1)} p_j}^{(1)} \neq 0\}$ ,  $q^{(1)} = (q'_2, q'_3, \dots, q'_t)$ . By our assumption and the commutator relations for  $\mathcal{L}[G]$ , we see that  $b_{q^{(1)} p_j}^{(1)} \neq 0$ , hence  $Q^{(1)} \neq \emptyset$ . Moreover,  $q^{(1)}$  is the unique minimum element in  $Q^{(1)}$ .

Now for  $s = 2, 3, \dots, t$ , we define recursively and prove by induction that

(i) Let  $u(s) := L_{q'_s - a} u(s-1)$ . Then

$$u(s) \equiv \sum_{p_i^{(s)} \in H_+, p_s, j_r \in \mathbb{Z}_+ a, |p^{(s)}| = t-s} b_{p^{(s)} p_j}^{(s)} I_{-p^{(s)}} I_{-a}^s I_{-p} L_{-j} v_h \pmod{V'_{t-s+1}} \text{ for some } b_{p^{(s)} p_j}^{(s)} \in \mathbb{F}^*.$$

(ii) Let  $Q^{(s)} := \{p^{(s)} \mid b_{p^{(s)} p_j}^{(s)} \neq 0\} \neq \emptyset$ . Moreover,  $q^{(s)} := (q'_{s+1}, q'_{s+2}, \dots, q'_t)$  is the unique minimum element in  $Q^{(s)}$ .

Now letting  $s = t$  and noting that  $u(t)$  is a weight vector, we get that  $0 \neq u(t) \in U(\mathcal{L}[G])u_0 \cap \widetilde{M}_a(h, h_I, c, c_I, c_{LI})$ , which gives (3.9) as required.  $\square$

## REFERENCES

- [ACKP] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, *Comm. Math. Phys.*, **117**(1988), 1-36.
- [B] Y. Billig, Representations of the twisted Heisenberg-Virasoro algebra at level zero, *Canad. Math. Bulletin*, **46**(2003), 529-537.
- [FO] M.A. Fabbri, F. Okoh, Representations of Virasoro-Heisenberg algebras and Virasoro-toroidal algebras, *Canad. J. Math.*, **51**(1999), no.3, 523-545.
- [HWZ] J. Hu, X. Wang, K. Zhao, Verma modules over generalized Virasoro algebras  $Vir[G]$ , *J. Pure Appl. Algebra*, **177**(2003), no.1, 61-69.
- [JJ] Q. Jiang, C. Jiang, Representations of the twisted Heisenberg-Virasoro algebra and the full toroidal Lie algebras, *Algebra Colloq.*, accepted.
- [LJ] D. Liu, C. Jiang, The generalized Heisenberg-Virasoro algebra, preprint (arXiv:math.RT/ 0510545).
- [SJ] R. Shen, C. Jiang, Derivation algebra and automorphism group of the twisted Heisenberg-Virasoro algebra, preprint.
- [WZ] X. Wang, K. Zhao, Verma modules over the Virasoro-like algebra, *J. Aust. Math. Soc.*, in press.
- [XLT] M. Xue, W. Lin, S. Tan, Central extension, derivations and automorphism group for Lie algebras arising from the 2-dimensional torus, *Journal of Lie Theory*, **16**(2005), 139-153.